On Extensibilities of Interconnection Networks

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Researchers have used nodes number to measure the extensibility of a topology. However, this metric is not very evident. In this paper, a specific metric called extensible density to measure the extensibilities of interconnection networks is introduced. Some topologies have high degree of extensibilities, but efficient parallel algorithms can apply only on a special subclass of these topologies. Furthermore, the concept of density to measure the applicable extent of parallel algorithms have been extended.

Keywords : Interconnection networks; Extensibilities; Parallel algorithms.

INTRODUCTION

The aim of studying interconnection networks and their combinatorial properties is to find good topologies for massively parallel computing. Ideally, a topology should have the following features: (1) efficient communication, (2) low hardware cost, (3) potentialities for efficient applications, (4) fault tolerance, (5) extensibility. Many metrics have been proposed to measure the above features. For example, diameter is the maximum distance of each pair of nodes in a graph. So, it can be used to measure the maximum communication delay[1]. Degree, link number, page number, cutwidth can be used to measure the hardware cost[2]. Bisection width, embedding capabilities can be used to measure the potentialities for efficient applications[1]. Connectivity, k-wide diameter, fault diameter can be used to measure the capabilities of fault tolerance[3].

In this paper, the term extensibility is used to represent the capability of a topology to interconnect a parallel system with a flexible size. In fact, many authors have noticed that extensibility is an important feature of interconnection networks[(4, 5)]. Researchers have used nodes number to measure the extensibility of a topology[1]. For example, the nodes number of hypercubes is 2^n. However, nodes number metric is not very evident. For example, the nodes number of Fibonacci tree is F_n, nth Fibonacci numbers[6]. The nodes number of mesh of trees[1] its $3\,(2^n)^2 - 2\,(2^n)$. How can their extensibilities be composed by the nodes numbers? In this paper, a specific metric called extensible density is introduced to measure the extensibilities of interconnection networks.

On the other hand, some topologies have high degree of extensibilities, but efficient parallel algorithms can apply only on a special subclass of these topologies[7]. Furthermore, the concept of density can be extended to measure the applicable extent of parallel algorithms.

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EXTENSIBLE DENSITY

Any interconnection network can be represented by an undirected graph $G\,(V, E)$ where V is the vertex set and E is the edge set. A topology, such as hypercubes, is a set of graphs. For a topology φ, the extensible domain of φ, denoted by *ext_dmn*(φ), is defined as $\{n|$ if there exists a graph $G_1\,(V_1, E_1) \in \varphi$ and $|V_1| = n\}$. For example, the extensible domain of hypercubes is $\{2^n|\, n \geq 0\}$. For a topology φ, a metric called extensible density is introduced which is defined as $|ext\_dmn\,(\varphi)|$ as $N \to \infty$, to measure its extensibility. Extensible density of a topology can be regarded as the number of different system sizes from 1 to N by the topology.

The primitive topologies, such as linear arrays, complete graphs, trees can be interconnected by arbitrary system size[1], so we can state the following lemmas:

Lemma 1. Extensible density of complete graphs is N.

Lemma 2. Extensible density of linear arrays is N.

Lemma 3. Extensible density of trees is N.

The minimum size[1] of rings is 3 hence.

Lemma 4. Extensible density of rings is N-2.

Although the extensible density of three is N, the subclasses of trees, such as full binary trees[1], full k-ary trees[1]. Fibonacci trees[6], have different extensible densities. The extensible domain of full binary trees is $\{2^n - 1|\, n \geq 1\}$. Considering $2^n - 1 = N$, $n = \log\,(N + 1)$ gives the following Lemmas,

Lemma 5. Extensible density of full binary trees is $\log\,(N + 1) \approx \log N$.

The extensible domain of full k-ary trees is $\{k^n - 1|\ n \geq 1\}$. Let $k^n - 1 = N$, $n = \dfrac{\log\,(N+1)}{\log k}$, we thus have

Lemma 6. Extensible density of full k-ary trees is $\dfrac{\log\,(N+1)}{\log k} \approx \dfrac{\log N}{\log k}$

Extensible domain of Fibonacci trees is $\{F_n$, nth Fibonacci number $\mid n \geq 1\} = \left\{ \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right) \mid n \geq 1 \right\}$

Lemma 7. Extensible density of Fibonacci trees is near 1.67228 + 1.44042 log N.

Proof.

Let $\frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right) = N$,

$$\therefore \left(\frac{1-\sqrt{5}}{2}\right)^n \to 0, \text{ as } n \to \infty$$

$$\therefore n \approx \frac{\log \sqrt{5} N}{\log \left(\frac{1+\sqrt{5}}{2}\right)} \approx 1.67228 + 1.44042 \log N \quad \text{QED}$$

Degree four chordal rings and (degree three) chordal rings are variations of rings[8]. The extensible domain of (degree three) chordal rings is $\{2n \mid n \geq 2\}$.

Lemma 8. Extensible density of chordal rings is 0.5 $N - 1$..

Proof. Let $2n = N$, $n = 0.5 N$.

$\therefore 2 \notin$ the extensible domain of (degree three) chordal rings,

$\therefore$ extensible density of chordal rings is 0.5 $N - 1$. QED

Extensible domain of degree four chordal rings is $\{n \mid n \geq 5\}$. Hence it gives the following lemma.

Lemma 9. Extensible density of degree four chordal rings is $N - 4$.

Meshes can be interconnected by arbitrary system size[1], thus the following lemma can be stated.

Lemma 10. Extensible density of meshes is N.

However, if the system size is a prime number, a mesh is reduced to a linear array. In fact, many parallel algorithms have devised on square meshes which is a 2-D mesh with the same row size and column size [2]. Clearly, extensible domain of square meshes is $\{n^2 \mid n \geq 1\}$. Let $n^2 = N$, $n = \sqrt{N}$. Which gives the following lemma.

Lemma 11. Extensible density of square meshes is $\sqrt{N}$

ILLIAC networks have the same extensible domain with square meshes[7] which gives

Lemma 12. Extensible density of ILLIAC networks is $\sqrt{N}$

X-trees are composition of full binary trees and linear arrays. Pyramids, multigrids, hypertrees are variation of X-trees[1]. X-trees and hypertrees have the same extensible domain with full binary trees, so we have

Lemma 13. Extensible density of X-trees is log $(N + 1) \approx \log N$.

Lemma 14. Extensible density of hypertrees is log $(N + 1) \approx \log N$.

Extensible domain of pyramids is $\left\{\frac{4^n - 1}{3} \mid n \geq 1\right\}$

Lemma 15. Extensible density of pyramids is $\frac{\log (3N + 1)}{2} \approx 0.79248 + 0.5 \log N$.

Proof. Let $\frac{4^n - 1}{3} = N$, $n = \frac{\log (3N+1)}{\log 4} \approx \frac{\log 3 + \log N}{2}$. QED

Multigrids have the same extensible domain with pyramids, we thus have

Lemma 16. Extensible density of multigrids is $\frac{\log (3N + 1)}{2} \approx 0.79248 + 0.5 \log N$.

Extensible domain of mesh of trees[1] is $\{3 (2^n)^2 - 2 (2^n) \mid n \geq 1\}$.

Lemma 17. Extensible density of mesh of trees is log $\left(\frac{1 + \sqrt{1 + 3N}}{3}\right)$

$$\approx -\frac{1}{2} \log 3 + \frac{1}{2} \log N \approx -0.79248 + 0.5 \log N.$$

Proof. Let $3 (2^n)^2 - 2 (2^n) = N$

$$3 (2^n)^2 - 2 (2^n) - N = 0$$

$$2^n = \frac{2 + \sqrt{4 + 12N}}{2 * 3}$$

$$2^n = \frac{1 + \sqrt{1 + 3N}}{3}$$

$$n = \log \left(\frac{1 + \sqrt{1 + 3N}}{3}\right) \approx \log \left(\frac{\sqrt{3N}}{3}\right) = -\frac{1}{2} \log 3 + \frac{1}{2} \log N$$

$$\approx -0.79248 + 0.5 \log N$$

Hypercubes are popular interconnection networks. Extensible domain of hypercubes is $\{2^n \mid n \geq 0\}$.

Lemma 18. Extensible density of hypercubes is $1 + \log N$.

Proof. Let $2^n = N$

$$n = \log N$$

Observe that $n = 0$ is feasible,

Extensible density of hypercubes is $1 + \log N$. QED

There are many variations of hypercubes. Bounded degree hypercubic networks includes de Bruijn networks, shuffle exchange networks CCC and butterfly networks[1].

De Bruijn networks and shuffle exchange networks have the same extensible domain, $\{2^n \mid n \geq 0\}$, with hypercube. This gives

Lemma 19. Extensible density of de Bruijn networks is $1 + \log N$.

Lemma 20. Extensible density of shuffle exchange networks is $1 + \log N$.

CCC and butterfly networks have the same extensible domain $\{n2^n | n \geq 1\}$. Let $n2^n = N$. We can choose $n' = \log N - \log \log N$, $n' 2^{n'} = (\log N - \log \log N) \dfrac{N}{\log N} = N - N\dfrac{\log \log N}{\log N} \leq N$. Thus, $\log N - \log \log N$ is lower bound for n. On the other hand, let $n' = \log N$, $n' 2^{n'} = N \log N \geq N$. Thus, $\log N$ is upper bound for n.

Therefore,

Lemma 21. Extensible density of CCC resides between $\log N - \log \log N$ and $\log N$.

Lemma 22. Extensible density of butterfly networks resides between $\log N - \log \log N$ and $\log N$.

Twisted cube family, including crossed cubes[9], twisted cubes[10], shuffle cubes[11], gain smaller diameters by changing some links on hypercubes. Crossed cubes and twisted cubes have the same extensible domain, $\{2^n | n \geq 0\}$, with hypercube. Thus

Lemma 23. Extensible density of crossed cubes is $1 + \log N$.

Lemma 24. Extensible density of twisted cubes is $1 + \log N$

Extensible domain of shuffle cubes is $\{2^{4n+2} | n \geq 1\}$ [13].

Lemma 25. Extensible density of shuffle cubes is $-0.5 + 0.25 \log N$.

Proof. Let $2^{4n+2} = N$

$$n = \frac{\log N - 2}{4} = -0.5 + 0.25 \log N \qquad \text{QED}$$

Incomplete hypercube family, including incomplete hypercubes[4] and supercubes[5], improve on extensibility of hypercubes. Incomplete hypercubes and supercubes can be used to interconnect arbitrary system size. Thus it gives

Lemma 26. Extensible density of incomplete hypercubes is N.

Lemma 27. Extensible density of supercubes is N.

HCN[3] and HFN[12] take hypercubic networks as basic modules and interconnect these modules with a complete graph. HFN and HCN have the same extensible domain $\{2^{2n} | n \geq 1\}$. Thus it gives,

Lemma 28. Extensible density of HFN is $0.5 \log N$.

Proof. Let $2^{2n} = N$, $n = 0.5 \log N$ QED

Lemma 29. Extensible density of HCN is $0.5 \log N$.

Extensible domain of k-ary n cubes is $\{k^n | n \geq 1\}$ [3].

Lemma 30. Extensible density of k-ary n cubes is $\dfrac{\log N}{\log k}$.

Proof. Let $k^n = N$, $n \log k = \log N$, $n = \dfrac{\log N}{\log k}$ QED

Fibonacci cubes have the same extensible domain with Fibonacci trees[10].

Lemma 31. Extensible density of Fibonacci cubes is near 1.67228 + 1.44042 log N.

Table 1 Extensible densities of topologies

Topology	Extensible Density
Complete graphs	N
Linear arrays	N
Trees	N
Meshes	N
Incomplete hypercubes	N
Supercubes	N
Rings	$N - 2$
Degree four chordal rings	$N - 4$
(Degree three) chordal rings	$0.5N - 1$
Square meshes	$\sqrt{N}$
ILLIAC networks	$\sqrt{N}$
Fibonacci trees	$\approx 1.67228 + 1.44042 \log N$
Fibonacci cubes	$\approx 1.67228 + 1.44042 \log N$
Hypercubes	$1 + \log N$
De Bruijn networks	$1 + \log N$
Shuffle exchange networks	$1 + \log N$
Crossed cubes	$1 + \log N$
Twisted cubes	$1 + \log N$
Full binary trees	$\log (N+1) \approx \log N$
X-trees	$\log (N+1) \approx \log N$
Hypertrees	$\log (N+1) \approx \log N$
CCC	$\log N \geq \text{density} \geq \log N - \log \log N$
Butterfly networks	$\log N \geq \text{density} \geq \log N - \log \log N$
Pyramids	$\approx 0.79248 + 0.5 \log N$
Multigrids	$\approx 0.79248 + 0.5 \log N$
HFN	$0.5 \log N$
HCN	$0.5 \log N$
Mesh of trees	$\approx -0.79248 + 0.5 \log N$
Shuffle cubes	$-0.5 + 0.25 \log N$
k-ary trees	$\dfrac{\log (N+1)}{\log k} \approx \dfrac{\log N}{\log k}$
k-ary $n$cubes	$\dfrac{\log N}{\log k}$

APPLICABLE DENSITY

In this section, we extend the concept of density to measure the applicable extent of parallel algorithms. For a parallel algorithm A, the applicable domain of A, denoted by $app\_dmn(A)$, is defined as $\{n |$ if the system size n is applicable for $A\}$. For a parallel algorithm A, a new metric called applicable density, defined as $|ext\_dmn(A)|$ as $N \to \infty$, can be used to measure its applicable extent.

As stated above, meshes can be applied to interconnect arbitrary system size. However, a mesh is reduced to a linear array as the system size is a prime number[2]. Thus, in fact, many algorithms can apply only on a subclass of meshes which density is less than N. For example, if an algorithm is restricted to apply on square meshes, its applicable density $\sqrt{N}$. If an algorithm can apply only on $2^n \times 2^n$ meshes, its applicable domain is $\{2^{2n} | n \geq 1\}$. Let $2^{2n} + N$, its applicable density is $n = 0.5 \log N$. This density is sparser than the density of hypercubes.

CONCLUSION

Researchers have used nodes number to measure the extensibility of a topology. Unfortunately, it is not very evident. For example, extensibilities between Fibonacci trees and mesh of trees cannot be directly compared by the nodes numbers. In this paper, a specific metric has been proposed, extensible density to measure the extensibilities of interconnection networks. The extensible densities of various interconnection networks have been investigated. In Table 1, these topologies are listed and their extensible densities. It can be observed from the table that the extensible density of Fibonacci tree is near 1.67228 + 1.44042 log *N* and the extensible density of mesh of trees is near -0.79248 + 0.5 log *N*. Evidently, by extensible densities, Fibonacci trees is superior to mesh of trees in extensibility. The concept of density can be extended to measure the applicable extent of parallel algorithms. How to improve the extensible density of various interconnection networks without losing their attractive properties, such as efficient parallel algorithms, low hardware cost, efficient communication, fauld tolerance, appear interesting.

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