Analysis of Arbitrarily Shaped Plates using Curved Quadrilateral Thin Element and Blending Functions for Mapping

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Analysis of arbitrary plates using curved quadrilateral twelve degree of freedom (dof) Kirchhoff plate element was presented by Meghre and Kadam. The approach consisted of mapping entire plate in a parent 2 ×2 ξ, η region using twelve geometry nodes on plate boundaries. This paper adopts the element published earlier. However, geometry mapping of the plate is achieved by specifying parametric equations of the four sides along with linear and bilinear interpolation functions. This method maps the boundaries exactly and eliminates slight inherent errors in twelve node mapping method. The discretisation of plate into elements is achieved by lines of constant ξ, η mapped on actual plate. Results of worked out examples are presented and compared with available results.

Keywords: Finite element; Thin plate; Arbitrary shape; Quadrilateral element; Blending functions

INTRODUCTION

Analysis of an arbitrarily shaped thin plate using classical approach of solving fourth order differential equation is a difficult task. Researchers have suggested alternate numerical and analytical methods to deal with arbitrary plates. These are mentioned as curvilinear finite difference method[1], Galerkin method[2], differential quadrature method[3], finite difference energy method[4,5], Rayleigh-Ritz method using pb2 functions[6]. The finite element method and the finite strip method based on Mindlin plate theory require C° continuity in displacement field. A large family of such elements is available. An isoparametric spline finite strip is developed by Au and Cheung[7] for general plate. The element based on thin plate theory requires C^1 continuity making it difficult to develop element with curved edges for general plates. It seems that, Li, Cheung and Tham[8] were the first to present general thin plate analysis using spline finite strip based on thin plate theory. Fan and Luah[9] presented nine noded curved quadrilateral thin element. Later, Meghre and Kadam[10] presented four noded, twelve degree of freedom (dof), curved quadrilateral element for analysis of general plate.

This paper uses an element presented by Meghre and Kadam[10], but adopts different method of mapping. In earlier paper, twelve geometry nodes on plate boundaries along with serendipity shape functions for geometry transformation were used. There are four geometry nodes on each side. Hence, cubic parabolic curved edge can be correctly modelled. But, curves like ellipse, arc of circle, quartic and quintic parabolae can not be modelled properly. There are inherent errors, though small, between the actual and the mapped boundary. Li, Cheung and Tham[8] suggested use of blending functions to eliminate such errors in mapping. The present paper, therefore, uses parametric equations of four sides of the plate and then obtains co-ordinate transformations within plate (and hence, within an element) through use of linear and bilinear interpolation functions.

The element has four displacement nodes. Each node is associated with three displacement parameters. The first and second derivatives of displacement are initially obtained as referred to ξ, η system and then transformed to orthogonal cartesian system using chain rule. Numerical integration is adopted to evaluate element stiffness matrix and load vector. The results of some examples are presented and compared with available results where possible.

FORMULATION

Figure 1 shows an arbitrary quadrilateral plate mapped into a ξ, η plane in the region $(-1 \le \xi \le 1)$ and $(-1 \le \eta \le 1)$. The lines of constant ξ and η are initially established on parent region. These lines mapped on actual plate discretise the plate into elements. These lines need not be equispaced. A typical element is bound in the region

$$\left.\begin{array}{ll} \xi_i \le \xi \le \xi_{i+1} & (0 \le r \le 1) \\ \eta_j \le \eta \le \eta_{j+1} & (0 \le s \le 1) \\ \text{with} & \\ r = (\xi - \xi_i)/a; & s = (\eta - \eta_j)/b \\ \text{and} & \\ a = (\xi_{i+1} - \xi_i); & b = (\eta_{j+1} - \eta_j) \end{array}\right\} \quad (1)$$

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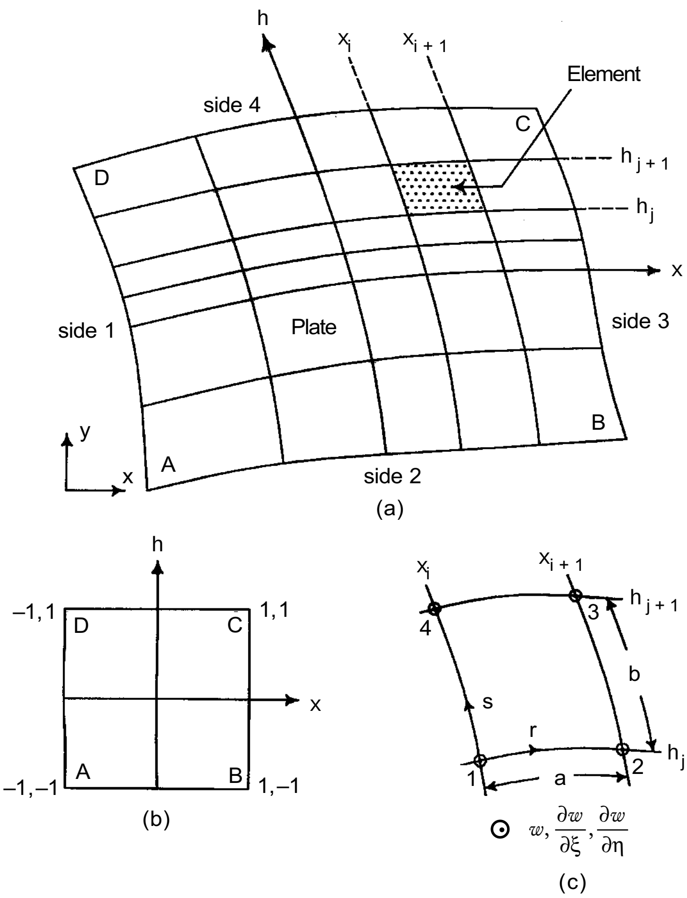

Figure 1 (a) actual plate; (b) parent region; and (c) curved element

Let parametric equations of four sides of the plate be written as

$$\left.\begin{array}{l} \left.\begin{array}{l} x = X_i(\eta) \\ y = Y_i(\eta) \end{array}\right\} ; \quad i = 1 \text{ and } 3 \\ \text{and} \\ \left.\begin{array}{l} x = X_i(\xi) \\ y = Y_i(\xi) \end{array}\right\} ; \quad i = 2 \text{ and } 4 \end{array}\right\} \qquad (2)$$

in which, i indicates number of side as shown in Figure 1.

Linear and bilinear functions required for transformation are

$$\left.\begin{array}{ll} L_1(\xi) = \frac{1}{2}(1-\xi), & L_2(\xi) = \frac{1}{2}(1+\xi) \\ L_1(\eta) = \frac{1}{2}(1-\eta), & L_2(\eta) = \frac{1}{2}(1+\eta) \\ \text{and} & \\ M_1(\xi,\eta) = L_1(\xi)\, L_1(\eta) & \\ M_2(\xi,\eta) = L_2(\xi)\, L_1(\eta) & \\ M_3(\xi,\eta) = L_2(\xi)\, L_2(\eta) & \\ M_4(\xi,\eta) = L_1(\xi)\, L_2(\eta) & \end{array}\right\} \qquad (3)$$

Then, co-ordinates x and y of a point P on plate are related to ξ and η as[8]

$$\left.\begin{array}{l} x(\xi,\eta) = P_{1x} + P_{2x} - P_{3x} \\ y(\xi,\eta) = P_{1y} + P_{2y} - P_{3y} \\ \text{in which} \\ P_{1x} = X_1(\eta)\, L_1(\xi) + X_3(\eta)\, L_2(\xi) \\ P_{2x} = X_2(\xi)\, L_1(\eta) + X_4(\xi)\, L_2(\eta) \\ P_{3x} = x_A M_1 + x_B M_2 + x_C M_3 + x_D M_4 \end{array}\right\} \qquad (4)$$

The expression for y can be obtained similarly by replacing x and X with y and Y, respectively. A, B, C and D are corner points at $\xi = \eta = \pm 1$ as shown in Figure 1.

The element displacement vector is defined as

$$\{\delta_e\} = \left\{ w_1 \left(\frac{\partial w}{\partial \xi}\right)_1 \left(\frac{\partial w}{\partial \eta}\right)_1 w_2 \text{ -----} \left(\frac{\partial w}{\partial \eta}\right)_4 \right\}^T \qquad (5)$$

The displacement w of a middle plane point within element is expressed as

$$w = [N][T]\{\delta_e\} \qquad (6)$$

in which, $[N]$ is a matrix of shape functions and $[T]$, a diagonal matrix of size 12 × 12.

$$\left.\begin{array}{l} [N] = [N_1\ N_2\ N_3 \ldots\ldots\ldots N_{12}] \\ [T] = \text{diagonal} \lceil 1\ a\ b\ 1\ a\ b\ 1\ a\ b\ 1\ a\ b \rfloor \end{array}\right\} \qquad (7)$$

The shape functions $N_i(r, s)$ based on twelve term polynomial are given in Meghre and Kadam[10].

A matrix of first and second derivatives of w with ξ, η is defined as

$$\{\chi^*\} = \left\{ \frac{\partial^2 w}{\partial \xi^2} \quad \frac{\partial^2 w}{\partial \eta^2} \quad 2\frac{\partial^2 w}{\partial \xi \partial \eta} \quad \frac{\partial w}{\partial \xi} \quad \frac{\partial w}{\partial \eta} \right\} \qquad (8)$$

From equation (1)

$$\frac{\partial w}{\partial \xi} = \frac{1}{a}\frac{\partial w}{\partial r}; \qquad \frac{\partial^2 w}{\partial \xi^2} = \frac{1}{a^2}\frac{\partial^2 w}{\partial r^2} \text{ etc} \qquad (9)$$

Using equations (5), (6), (8) and (9), $\{\chi^*\}$ is expressed in terms of $\{\delta_e\}$ as

$$\{\chi^*\} = [B^*][T]\{\delta_e\} \qquad (10)$$

in which, $[B^*]$ is a relation matrix containing derivatives of N.

The bending strains

$$\{\chi\} = \left\{ -\frac{\partial^2 w}{\partial x^2} \quad -\frac{\partial^2 w}{\partial y^2} \quad -2\frac{\partial^2 w}{\partial x \partial y} \right\}^T \qquad (11)$$

are related to $\{\chi^*\}$ by a relation matrix as

$$\{\chi\} = (-)\,[R]\left\{\chi^*\right\} \tag{12}$$

Matrix $[R]$ is derived by chain rule of transformation and is given in Meghre and Kadam[10].

Combining equations (10) and (12)

$$\{\chi\} = (-)\,[R]\left\{B^*\right\}[T]\{\delta_e\} = [B]\{\delta_e\} \tag{13}$$

The moments are related to bending strains as

$$\{M\} = [D]\,\{\chi\} \tag{14}$$

in which, $[D]$ is an elasticity matrix for thin plate.

Total potential energy of an element subjected to load of intensity q/area is

$$\pi_e = \frac{1}{2}\int_A \{\chi\}^T\,[D]\{\chi\}\,\mathrm{d}A - \int_A w^T\,q\,\mathrm{d}A \tag{15}$$

Also

$$\pi_e = \frac{1}{2}\,\{\delta_e\}^T\,[k_e]\{\delta_e\} - \{\delta_e\}^T\,\{F_e\} \tag{16}$$

Substituting from equations (6) and (13) in equation (15) and comparing with equation (16)

$$\left.\begin{aligned} [k_e] &= \int_{\xi_i}^{\xi_{i+1}}\int_{\eta_j}^{\eta_{j+1}} [B]^T\,[D][B]\,J^*\,\mathrm{d}\xi\,\mathrm{d}\eta \\ \{F_e\} &= [T]^T\iint q\,[N]^T\,J^*\,\mathrm{d}\xi\,\mathrm{d}\eta \end{aligned}\right\} \tag{17}$$

These are obtained using 2 × 2 integration.

Thus,

$$\left.\begin{aligned} [k_e] &= \sum_{i=1}^{2}\sum_{j=1}^{2}\left(\frac{ab}{4}\right)[B]^T\,[D][B]\,W_i\,W_j\,J^* \\ \{F_e\} &= [T]^T\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\frac{qab}{4}\right)W_i\,W_j\,[N]^T\,J^* \end{aligned}\right\} \tag{18}$$

W_i, W_j are Gauss point weightages and J^* is determinant of Jacobian matrix.

The analysis of plate based on this formulation follows usual finite element process, which includes assembly of element stiffness and load matrices, solution of equations for unknown nodal displacements and calculation of stress resultants. In the program prepared, stress resultants calculated at Gauss locations are extrapolated to nodes for finding nodal averages.

NUMERICAL EXAMPLES

Example 1

A clamped elliptical plate (a/b = 1.5) subjected to uniform load is analysed. Full plate is mapped in 2 × 2 region. Hence, ξ, η lines cross symmetry lines normally. Quarter plate is analysed using proper boundary conditions. Numerical values adopted are a = 1500 mm, b = 1000 mm, h = 20 mm, $E = 2 \times 10^5$ N/mm^2, υ = 0.3 and q = 0.01 N/mm^2. The co-ordinates of corner C are x = 1200 mm and y = 600 mm.

The deflection and moment coefficients are defined as

$$\left.\begin{aligned} w &= \alpha w_0 \\ M_x &= \beta_x\left(w_0\,D/a^2\right) \\ M_y &= \beta_y\left(w_0\,D/b^2\right) \\ \text{and} \quad w_0 &= \frac{q}{D\left(24/a^4 + 24/b^4 + 16/a^2b^2\right)} \end{aligned}\right\} \tag{19}$$

Parametric equations for sides 3 and 4 are

$$\left.\begin{aligned} X_3(\eta) &= 1500\sqrt{1 - 0.36\,\eta^2} \\ Y_3(\eta) &= 600\,\eta \\ X_4(\xi) &= 1200\,\xi \\ Y_4(\xi) &= 1000\sqrt{1 - 0.64\,\xi^2} \end{aligned}\right\} \tag{20}$$

Equations for remaining two sides can be written similarly.

Table 1 shows results for important deflection and moments for different mesh sizes. The plots of deflection and moment coefficients along major axis (6 × 9 mesh) are shown in Figure 2. All results are in good agreement with the theoretical results.

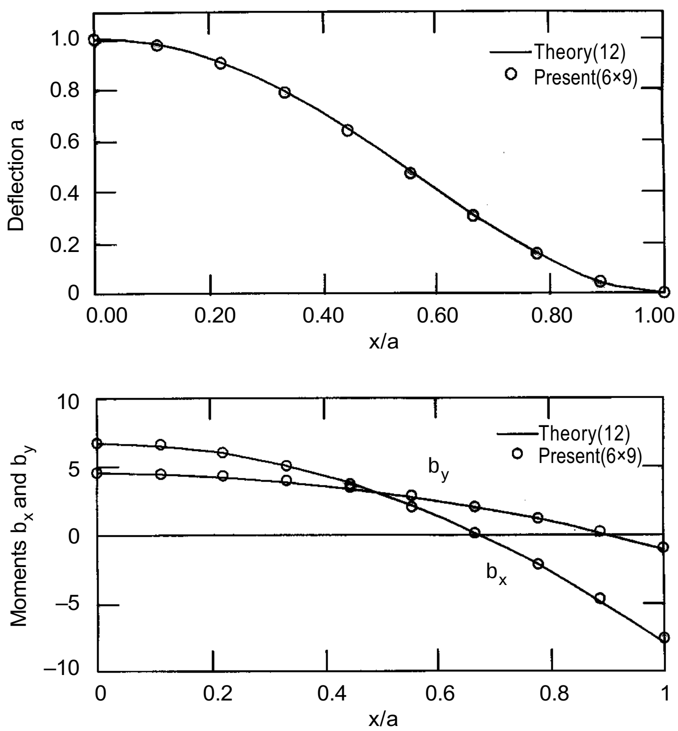

Figure 2 Example 1 — clamped elliptical plate — deflection and moments along major axis

Table 1 Example 1 — clamped elliptical plate (a/b = 1.5) under uniformly distributed load (υ = 0.3)

Mesh	Centre $x = y = 0$			$x = a, y = 0$		$x = 0, y = b$	
	α	β_x	β_y	β_x	β_y	β_x	β_y
3 × 5	0.9880	7.027	4.692	– 6.393	– 0.5145	– 4.684	– 7.5460
4 × 6	0.9928	6.903	4.622	– 7.018	– 0.7106	– 4.962	– 7.7320
5 × 8	0.9959	6.820	4.594	– 7.413	– 0.8542	– 5.122	– 7.8370
6 × 9	0.9971	6.790	4.575	– 7.568	– 0.9058	– 5.200	– 7.8840
Exact[12]	1.0000	6.700	4.533	– 8.000	– 1.0660	– 5.400	– 8.0000
Meghre and Kadam[10], 6 × 9	0.9873	6.875	4.541	– 7.242	– 0.9660	– 5.071	– 7.5120

Example 2

Figure 3 shows S-bridge with three straight supports. This bridge was earlier analysed by Cheung, Tham and Li[11] using spline finite strip (thin) and by Au and Cheung[7] using isoparametric Mindlin spline strip. Each of the curved sides is defined by four control points and hence is expressible by cubic parabola in ξ. The straight simple supports are located at $\xi = -1, 0$ and $+1$. The bridge is analysed using 6 divisions in η and 24 divisions in ξ direction. The spacing is constant in each direction. Numerical values of E = 10 000, $\upsilon = 0.15, h = 0.2$ and q = 1 are adopted. The parametric equations of the curved sides are

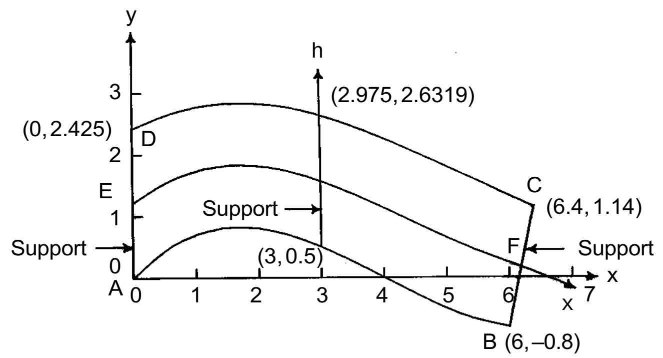

Figure 3 Example 2 — geometry of S-bridge

$$\left.\begin{array}{l}\text{Side 2 (AB)}\\ \quad X_2(\xi) = 3\xi + 3\\ \quad Y_2(\xi) = 0.9\xi^3 - 0.9\xi^2 - 1.3\xi + 0.5\\ \text{Side 4 (CD)}\\ \quad X_4(\xi) = 0.225\xi^3 + 0.225\xi^2 + 2.975\xi + 2.975\\ \quad Y_4(\xi) = 0.2475\xi^3 - 0.8494\xi^2 - 0.89\xi + 2.6319\end{array}\right\} \quad (21)$$

The results of deflection w and moments M_x and M_y on $\eta = 0$ line (EF) are plotted in Figure 4. Note that $\eta = 0$ line does not coincide with x-axis. Hence, x co-ordinates of the concerned nodes are used for plotting. The results are compared with the

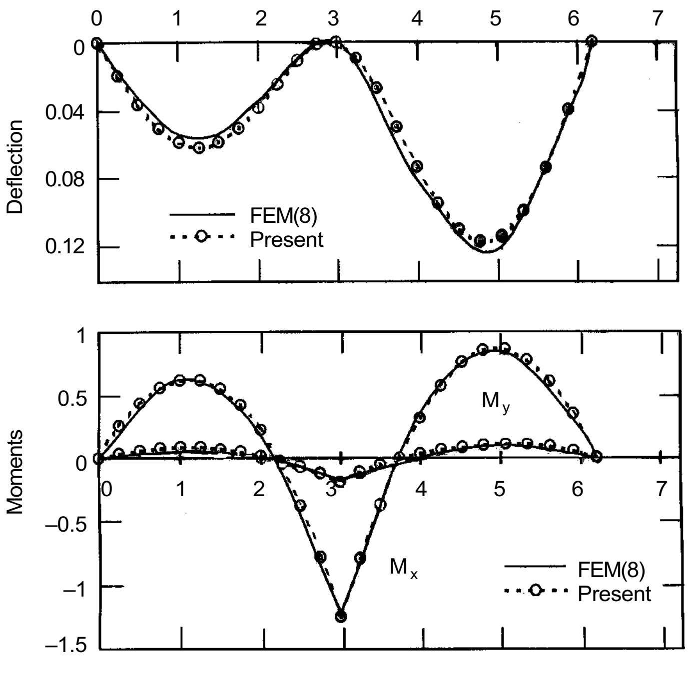

Figure 4 Example 2 — deflection and moments on line EF (η = 0)

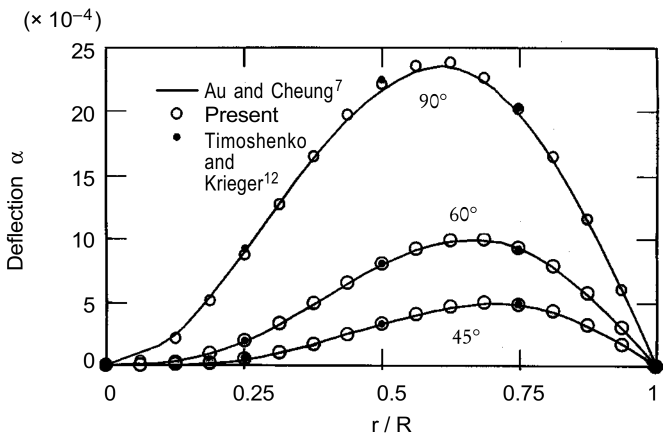

Figure 5 Example 3 — annular sector plate — deflection along central radial line

results of finite element method as reported by Au and Cheung[7]. Figure 4 shows that the results of present method are in reasonably good agreement with the FEM results.

Example 3

Although formulation of the method is developed for quadrilateral plate, it is possible to analyse a plate with three sides. In this case, one of the four sides degenerates into a single point. To demonstrate this feature, annular sector plates with included angles of $45^\circ, 60^\circ$ and 90°, simply supported on all sides, carrying uniform load are analysed. Full plate is mapped in 2×2 region and half plate is analysed using 16×8 mesh. The plots of deflection coefficients on central radial line are shown in Figure 5. These are compared with Saadatpour and Azhari[2] and analytical solution[12] showing good agreement. The α coefficient is defined as $w = \alpha \, qR^4 / D$.

CONCLUDING REMARKS

A four noded curved quadrilateral plate bending element based on thin plate theory is presented. The element is proposed to be used for the analysis of an arbitrarily shaped quadrilateral plate with straight or curved boundaries. Entire plate is mapped in 2×2 region using blending functions. This method of mapping exactly maps the boundaries and slight errors inherent in mapping by 12 geometry nodes are eliminated. The discretisation results in elements which are bound by lines of constant ξ and η. The results of three examples presented clearly demonstrate the validity of the formulation and applicability of the method to general plates.

It is mentioned that, the parametric equations of the boundaries change from problem to problem. These expressions and also expressions for derivatives like $\partial x / \partial \xi$, $\partial^2 x / \partial \xi^2$ etc are to be changed in the source program. These changes can not be effected through external numerical data.

It is hoped that the work presented will give impetus to further work in the field.

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