Band Limited Signal Interpolation using Eulerian Filters

M Dash, *Non-member*
L Samantaray, *Non-member*
Dr R Panda, *Member*

This paper presents a linear mathematical analysis for the interpolation of band limited signals using Eulerian filters. Some useful results on the derivatives of the sigmoid function, used as neuron activation function in the subject field of artificial neural networks (ANN) are presented. These results establish the relations of the coefficients of different derivatives of a sigmoid function to the standard Eulerian numbers. Interesting results showing factorisation of the transfer functions of higher order derivative filters are also produced. Global frequency response of such higher order filters has been shown to interpret the quality of interpolation in frequency domain. An experiment has been carried out to interpolate a discrete sequence of length 33 into a sequence of length 257. The present scheme is time efficient as it requires only three multiplications and additions per sample point.

Keywords : Signal processing; Eulerian numbers; Multiresolution technique

INTRODUCTION

Interpolation of band limited signals play an important role in many digital signal and image processing applications. Estimating intermediate values from discrete samples (data) to provide a continuous display or representation of the original signal is very important in the fields of digital signal/image processing[1, 2].

The proposed interpolation scheme is a two stage process. (i) The signal is passed through a pre-filter. (ii) The signal is reconstructed by interpolating the sampled data with a post filter. Before the second stage, the signal is up sampled by a factor of *m*, *ie*, resolution factor to facilitate signal interpolation and representation.

Some interesting results are obtained in connection with the derivatives of the 1-dimensional sigmoid function :

$$y = \frac{1}{1 + e^{-\omega x}} \tag{1}$$

where *w* is the weight. This sigmoid function is extensively used in artificial neural networks derived from its utility in Bayesian estimation of classification of probabilities[3, 4]. The sigmoid function with different values of the weight *w* is shown in Figure 1.

The coefficients of these derivatives are related to the Eulerian numbers. This has motivated us to derive different symmetrical filter transfer functions using these relations. Later these filters are used for the continuous representation of discrete signals through interpolative signal reconstruction with an expansion factor *m*. The present paper also provides interesting results showing factorisation of the transfer function of higher order derivative filters. Global frequency response of the proposed interpolative signal reconstruction scheme has been shown in the frequency domain for better interpretation of the quality of interpolation.

M Dash, L Samantaray and Dr R Panda are with the Department of Electronics & Telecommunication Engineering, University College of Engineering, Burla 768 018.

This paper was received on December 12, 2001. Written discussion on the paper will be entertained till September 30, 2003.

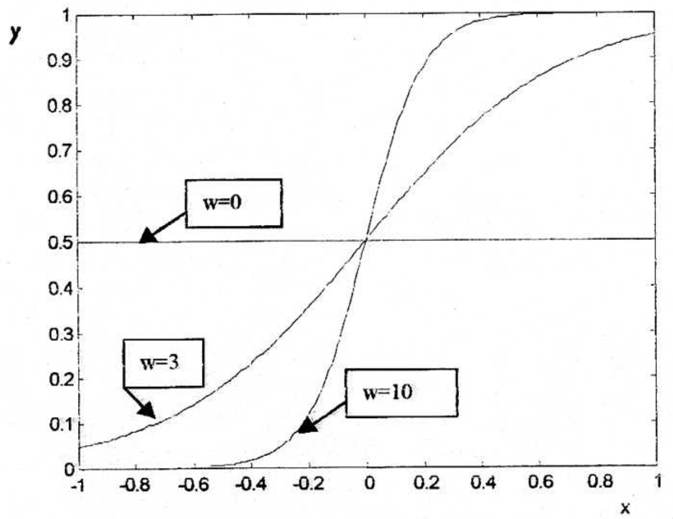

Figure 1 Sigmoid function with three different weights

MATHEMATICAL ANALYSIS

The first derivative of *y* is given as :

$$\frac{dy}{dx} = \rho^{(1)} = w\frac{e^{-\omega x}}{(1 + e^{-\omega x})^2} = w\,y\,(1 - y) \tag{2}$$

where $\rho^{(1)}$ indicates the first derivative function.

The second derivative of *y* is :

$$\frac{d}{dx}\left[y^k (1-y)^l\right] = wky^k (1-y)^{l-1} - wly^{k+1} (1-y)^l \tag{3}$$

The weight '*w*' can be eliminated from right hand side of equation 3 by arithmetic manipulations.

$$\rho^{(n)} = \sum_{k=1}^{n} (-1)^{k-1} c_k^{(n)} y^k (1 - y)^{n+1-k} \tag{4}$$

where $\rho^{(n)}$ represents *n*th derivative function and $C_k^{(n)}$ are Eulerian numbers. Table 1 shows the Eulerian numbers for different higher order derivatives.

Table 1 The coefficients $C_k^{(n)}$

Derivative Number (n)	K								
	1	2	3	4	5	6	7	8	9
1	1								
3	1	4	1						
5	1	26	66	26	1				
7	1	120	1191	2416	1191	120	1		
9	1	502	14608	88234	156190	88234	14608	502	1

Table 2 Filter transfer function $(H^{(n)})^{-1}z$

Derivative Number (n)	$(H^{(n)})^{-1}(z)$
1	1
3	$1/(z+4+z^{-1})$
5	$1/(z^2+26z+66+26z^{-1}+z^{-2})$
7	$1/(z^3+120z^2+1191z+2416+1191z^{-1}+120z^{-2}+z^{-3})$

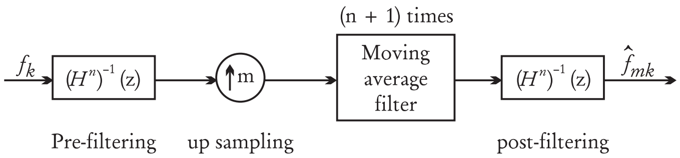

Figure 2 Block diagram of an interpolator using Eulerian filters

In the proposed interpolation scheme, symmetrical I I R filters derived from the above derivative functions are used in the first stage as shown in Figure 2. Different filter transfer functions for different odd values of 'n' are shown in Table 2.

From the combinatorial theory, the Eulerian numbers $C_{l-|j|}$ can be written as

$$C_{l-|j|} = \sum_{i=0}^{l-j-1} (-1)^i \binom{n+1}{i} (l-j-i)^n \qquad (5)$$

where $l = \dfrac{n+1}{2}$ and j varies from $-l+1$ to $l-1$.

In this scheme, the interpolated function is given by :

$$\hat{f}(x) = \sum_{K=0}^{N-1} s_k \, \rho^{(n)} (x - k) \qquad (6)$$

where s_k is the coefficient of interpolation, N, the number of data points; and $\rho^{(n)}$, the sigmoid derivative functions of order 'n'.

From linear mathematical analysis, equation (6) can also be written as :

$$\hat{f}_k = \sum_{j-l+1}^{l-1} s_{k-j} \; C_{l-|j|} \qquad (7)$$

where C_k's are the elements of the system matrix generated by sigmoid derivative functions and $l = \dfrac{n+1}{2}$.

By introducing b_k for $k = 1, 2, 3, ..., N$ and q_i for $i = 1, 2, ..., l-1$, equation (7) becomes

$$\hat{f}_k = q_1 b_{k-l-2} + \ldots + q_{l-2} b_{k-1} + q_{l-1} b_k + q_{l-2} b_{k+1} + \ldots + q_l b_{k+l-2} \qquad (8)$$

where b_k is

$$b_k = s_{k-1} + \gamma s_k + s_{k+1} \qquad (9)$$

Substituting Equation (9) in Equation (8) and equating the s_k coefficients term by term of equation (7), gives a recursive relation of the form

$$q_i = c_i - \gamma q_{i-1} - q_{i-2} \quad \text{for } i = 3, 4, \ldots, l-1 \qquad (10)$$

with $q_1 = c_1 = 1$ and $q_2 = c_2 - \gamma$. In general, q_k can be expressed as

$$q_k = \sum_{i=0}^{k-1} (-1)^i \gamma^i \sum_{j=0}^{j \le \frac{i-l-2}{2}} (-i)^j \binom{i+j}{j} c_{l-2j-i-1} \qquad (11)$$

Equation (11) being a polynomial in γ of degree $(l-1)$ can be written as :

$$g_i \gamma^{l-1} - g_2 \gamma^{l-2} + g_3 \gamma^{l-3} - \ldots, g_l = 0 \qquad (12)$$

Equating the coefficients γ^{l-K} of equation (12) with that of equation (11), results in a recursive relation for g_k as

Table 3 Roots γ_i of the polynomial

Order (n)	C_k	g_k	γ_i
5	66	64	3.128813
	26	26	72.871245
	1	1	
7	2416	2176	2.403464
	1191	1188	8.282187
	120	120	109.314356
	1	1	
9	156190	126976	2.252749
	88234	86728	5.158374
	14608	14604	23.179267
	502	502	471.409632
	1	1	
11	15724248	11321344	2.173521
	9738114	9280208	3.946214
	2203488	2195344	11.230634
	152637	152632	60.006012
	2036	2036	1958.643638
	1	1	

$$g_k = c_k - \sum_{i=1}^{i \le \frac{k-1}{2}} \left(\frac{l-k+2}{i} \right) g_{k-2i} \quad \text{for} \quad 3 \le k \le l \tag{13}$$

The basic fundamentals of linear mathematical analysis[5], shows that equation (12) has $(l-1)$ real and positive roots γ_i $(i = 1, 2, 3, \ldots, l-1)$.

The coefficients C_k for $[K = l, \ldots, 2, 1]$ are tabulated in Table 1. The coefficients g_k for $[K = l, \ldots, 2, 1]$ are evaluated by using equation 13 and the roots of polynomial can be determined by any standard root finding subroutine or using MATLAB. The coefficients C_k, g_k and the roots γ_i are shown in Table 3.

With the roots γ_i displayed in Table 3; the higher order filter transfer functions can be expressed in the factored from as :

$$\left(H^{(n)}\right)^{-1}(z) = \frac{1}{\Pi_{i=1}^{l-1}\left(z + \gamma_i + z^{-1}\right)} \tag{14}$$

Thus, the generic symmetrical I I R filter $\left(H^{(n)}\right)^{-1}(z)$ can be decomposed into a cascade of elementary I I R filters $\left(\frac{1}{z + \gamma_i + z^{-1}}\right)$ with $i = 1, 2, \ldots, l-1$. Each elementary I I R filter can be implemented recursively.

The global frequency response of the proposed interpolator shown in Figure 2 is, therefore, given as :

$$H^{(n)}(\omega) = \prod_{i=1}^{l-1} \frac{1}{\gamma_i + 2\cos\omega m} \left(\frac{\sin(\omega m/2)}{(\omega m/2)} \right)^{n+1} \tag{15}$$

where m is the resolution factor.

The gain against frequency characteristics for different odd values of n with resolution 1, *ie* $m = 1$ are shown in Figure 3.

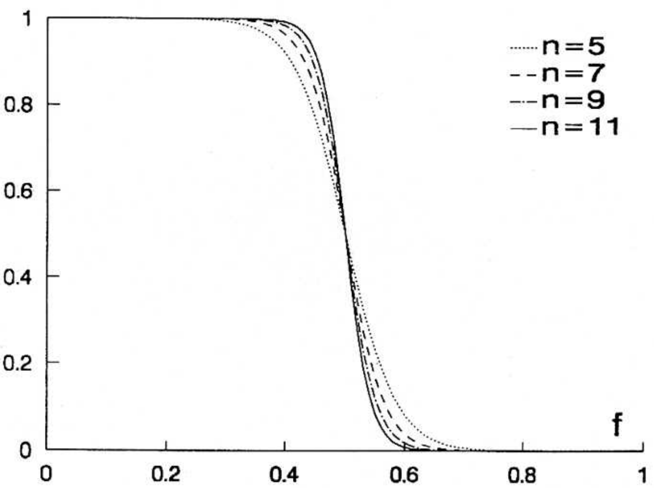

Figure 3 Gain against frequency characteristics of higher order filters

Table 4 Performance evaluation results

Test Function	1	2	3	4	5	6	7	8
N M S E [dB]	− 25.63	− 39.51	− 27.20	− 40.82	− 43.91	− 73.34	− 55.96	− 56.22

INTERPOLATION EXAMPLE

Since long signal interpolation has been an important area of research. An example to interpolate a sequence $f(m)$, $m = 0, 1, \ldots, N$, of length $(N+1)$ into a sequence of length $(PN+1)$ using the proposed method has been presented where P is a positive integer decided by the user. In general, the following functions are considered for such experiments. The functions are (i) $f(x) = \cos(5\pi x/32 + 1)\exp(-x/12)$; (ii) $f(x) = -0.5\cos(3\pi x/32)$; (iii) $f(x) = 2\cos(3.1\pi x/32 + 1) + \cos(6.8\pi x/32 - 2)$; (iv) $f(x) = x/32$; (v) $f(x) = \sqrt{(x+1)}$; (vi) $f(x) = 4 - (x-16)$^$2/64$; (vii) $f(x) = 1.7\sin(3\pi x/32)$; (viii) $f(x) = \log(1+x)$. These continuous functions are sampled at unit intervals in between $t = 0$ to $t = 32$. The authors have considered interpolation of a discrete sequence of length 33 into a sequence of length 257 for this particular experiment. The performance measure is

$$NMSE = 10\log_{10}\left[\frac{\sum_{x=0}^{256}\left[f\left(\frac{x}{8}\right) - y(x)\right]^2}{\sum_{x=0}^{256} f\left(\frac{x}{8}\right)^2}\right] \tag{16}$$

where NMSE is the normalised mean square error expressed in dB. The results are presented in Table 4.

CONCLUSION

In this paper, the authors have presented a mathematical analysis for the interpolation of band limited signals using higher degree Eulerian filters. This approach is more simple to derive filter transfer functions (Table 2). Factorisation of the denominator polynomial of these filters has been shown. It is essentially this part of the algorithm that enables easy implementation of such higher filters. From Figure 3, it is observed that the cut off rate and the bandwidth of the filter increase with increase in the order n. These higher order filters possess flat characteristics. Results presented in Table 4 reveal the suitability of the method for interpolation.

REFERENCES

1. J G Proakis and D G Manolakis. 'Digital Signal Processing : Principles, Algorithms and Applications.' *Prentice Hall of India Pvt Ltd*, New Delhi, 2000.

2. A K Jain. 'Fundamentals of Digital Image Processing.' *Prentice-Hall of India*, New Delhi, 1995.

3. R Lippmann. 'An Introduction to Computing with Neural Nets.' *IEEE ASSP Magazine*, vol 4, 1987, p 4.

4. S Grossberg. 'Neural Networks and Natural Intelligence.' *MIT Press*, Cambridge, 1988.

5. J B Scarborough. 'Numerical Mathematical Analysis.' *Oxford and IBH Publishing Co*, New Delhi, 1979.